Mathematical olympiad of Baltic Sea schools 2011
3rd year of upper secondary school

1. Which is bigger,
$$2^{4^{6^8}} \text{ or } 8^{6^{4^2}}?$$

2. Determine the greatest value of the function $f\colon \mathbb{R} \to \mathbb{R}$,
$$f(x) = \ln(\sin x + 2) + (\sin x)^3,$$
if it exists.

3. A magician of the circus is asked, what is the value of
$$\sqrt[71]{1\,004\,525\,211\,269\,079\,039\,999\,221\,534\,496\,697\,502\,180\,541\,686\,174\,722\,466\,474\,743}.$$
The name of the game is that the answer is an integer; this is something that everyone knows in advanve. The magician impresses the audience by answering almost immediately. The answer is checked and proved to be right. Afterwards it is revealed that the magician only paid attention to the last digit of the radicand and estimated the number of digits. What is the answer and how did the magician find that out?

4. Consider the rational number
$$q = \frac{1^1 \cdot 2^2 \cdot 3^3 \ldots 2011^{2011}}{1^{2011} \cdot 2^{2010} \cdot 3^{2009} \ldots 2011^1} \left(= \prod_{k=1}^{2011} \frac{k^k}{k^{2012-k}} \right).$$
Prove that, in fact, q is an integer.

5. Find points A, B, C, D and E are placed on the plane. Consider the connecting line segments $AB, AC, AD, AE, BC, \ldots, DE$. Show that the locations of the points can be chosen so that seven of the connecting line segments have length one. Prove that eight of the line segments cannot possibly have length one.